Brody Curves and Mean Dimension

arXiv:1110.6014

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February 14th

Mean Dimension Formula

$$\dim(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}) = 4\rho(\mathbb{C}P^1)$$

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- Space of Brody curves
- Mean dimension
- Mean energy

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Motivation

I am fascinated with infinite dimensional geometry.

ISSUES: The world is too wild.

- Are there good examples of infinite dimensional spaces?
- Which direction to explore?

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OUR ANSWERS

- Moduli spaces of geometric non-linear PDE's on non compact manifolds are good examples.
- Consider dynamical systems.
- Gromov's mean dimension opens the new world.

First Topic: Space of Brody Curves

$$\dim(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}) = 4\rho(\mathbb{C}P^1)$$

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What is a Brody Curve?

A Brody curve is a 1-Lipschitz holomorphic map from $\mathbb{C}$.

Definition

A holomorphic map $f : \mathbb{C} \to \mathbb{C}P^N$ is Brody, if

$$|df|(z) \leq 1$$

for any $z \in \mathbb{C}$.

Examples of Brody Curves: $\mathbb{C} \to \mathbb{C}P^1$

A Brody curve is a 1-Lipschitz holomorphic map.

- "Identitiy" map $i\colon z \mapsto [1 : z]$ is Brody:

$$|di|(z) = \frac{1}{\sqrt{\pi}(1+|z|^2)} \leq 1$$

- For every rational function $f\colon \mathbb{C} \to \mathbb{C}P^1$, there exists a constant C such that $|df|(z) \leq C$.
- For every elliptic function $g\colon \mathbb{C} \to \mathbb{C}P^1$, there exists a constant C such that $|dg|(z) \leq C$.
- Exponential map $\exp\colon z \mapsto [1 : \exp(z)]$ is Brody:

$$|d\exp|(z) = \frac{e^x}{\sqrt{\pi}(1+e^{2x})} \leq 1$$

NON-example of Brody Curves

- Identitiy map $i\colon z \mapsto z$ is Brody.
- Exponential map $\exp\colon z \mapsto \exp(z)$ is Brody.

But,

- the sum $i + \exp\colon z \mapsto z + \exp(z)$ is NOT Brody.

Brody condtion $|df|(z) \leq 1$ is extreamly non-linear:

$$|df|^2(z) = \frac{1}{4\pi}\Delta\left[\log\left(|f_0|^2 + \ldots |f_N|^2\right)\right]$$

Space of Brody Curves

Definition

The space of Brody curves is the set of all Brody curves:

$$\mathcal{M}(\mathbb{C}P^N) := \{f : \mathbb{C} \to \mathbb{C}P^N \mid \bar{\partial} f = 0 \text{ and } |df|(z) \leq 1\}$$

- Topology of uniform convergence on compact subsets

$$f_j \to f \iff f_j|_K \xrightarrow{C^0} f|_K \text{ for every compact } K \subset \mathbb{C}$$

The moduli space is infinite dimensional:

$$\dim(\mathcal{M}(\mathbb{C}P^N)) = \infty$$

Next Topic: Mean Dimension

$$\dim(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}) = 4\rho(\mathbb{C}P^1)$$

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What is Mean Dimension?

Mean dimension was introduced by Gromov as an analogue of dimension for dynamical systems.

“Dimension” for Dynamical Systems

As You Know

The N-dimensional sphere is N-dimensional: $\dim \mathbb{S}^N = N$

Dynamical System Version

The bi-infinite product of $\mathbb{S}^N$ and the shift action of $\mathbb{Z}$:

$$\mathbb{Z} \curvearrowright \cdots \times \mathbb{S}^N \times \mathbb{S}^N \times \mathbb{S}^N \times \cdots, \quad \{p_n\} \overset{k}{\mapsto} \{p_{n+k}\}$$

Then, its mean dimension is

$$\dim \left((\mathbb{S}^N)^{\mathbb{Z}} : \mathbb{Z} \right) = N$$

Mean Dimension: Intuitive Description

Intuitively speaking,

$$\dim\left((\mathbb{S}^N)^{\mathbb{Z}} : \mathbb{Z}\right) = \frac{\dim\left((\mathbb{S}^N)^{\mathbb{Z}}\right)}{|\mathbb{Z}|}$$

and

$$\dim\left((\mathbb{S}^N)^{\mathbb{Z}}\right) = \dim \mathbb{S}^N \times |\mathbb{Z}|$$

Therefore,

$$\dim\left((\mathbb{S}^N)^{\mathbb{Z}} : \mathbb{Z}\right) = \frac{\dim \mathbb{S}^N \times |\mathbb{Z}|}{|\mathbb{Z}|} = \dim \mathbb{S}^N$$

Mean Dimension: More Formal Description

Mean dimension is a topological invariant of compact metrizable spaces with continuous amenable group action.

G: Amenable Group $\curvearrowright$ Y: Compact Metrizable Space

$$\downarrow$$

$$\dim(Y : G) \in [0, \infty]$$

Its definition is quite similar to that of Topological Entropy.

Point counting : Dimension = Entropy : Mean dimension

Mean Dimension: Propaganda

Mean dimension is quite Gromovian.

- Its definition is easy.
- Its calculation seems impossible.
- It is a source of inspiration for future geometry.

Final Topic: Mean Energy

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We Need Mean Energy

Energy of a map is the L^2-norm of its derivative.

Energy of Brody curves can be infinite:

$$\int_{\mathbb{C}} |d \exp|^2 \, dz = \int_{\mathbb{C}} \left(\frac{e^x}{\sqrt{\pi}(1 + e^{2x})} \right)^2 dxdy = \infty$$

Mean Energy: Definition

Mean Energy is a renormalized energy of Brody curves.

Definition

For a Brody curve $f\colon \mathbb{C} \to \mathbb{C}P^N$, we define

$$\rho(f) := \lim_{R\to\infty} \frac{1}{\pi R^2} \left[\sup_{a\in\mathbb{C}} \int_{|z-a|\le R} |df|^2 \, dxdy\right]$$

- This limit always exists
- Always finite: $0 \le \rho(f) \le 1$
- Theorem: $0 \lneqq \rho(f) \lneqq 1$

Definition

$$\rho(\mathbb{C}P^N) := \sup\{\rho(f) \mid f \in \mathcal{M}(\mathbb{C}P^N)\}$$

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Final Confirmation

- $\mathcal{M}(\mathbb{C}P^N)$ is compact and metrizable.
- $\mathbb{C}$ acts continuously on $\mathcal{M}(\mathbb{C}P^N)$:

$$f(z) \overset{w}{\mapsto} f(z+w)$$

- Mean dimension is an invariant of compact metrizable spaces with continuous $\mathbb{C}$-action

Therefore, we can consider the mean dimension of $\mathcal{M}(\mathbb{C}P^N)$.

Main Theorem

Joint work with Masaki Tsukamoto

Theorem

$$2(N+1)\rho(\mathbb{C}P^N) \leq \dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C}) \leq 4N\rho(\mathbb{C}P^N)$$

Main Theorem

Joint work with Masaki Tsukamoto

Theorem

$$2(N+1)\rho(\mathbb{C}P^N) \le \dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \le 4N\rho(\mathbb{C}P^N)$$

Corollary ($N = 1$)

$$\dim(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}) = 4\rho(\mathbb{C}P^1)$$

One More Thing: Yang-Mills Gauge Theory

Theorem (Masaki Tsukamoto and SM)

There exist a constant d_0 and a countable subset $D \subset [d_0, \infty)$ such that

$$\dim_{loc}(\mathcal{M}_d : \mathbb{Z}) = 8\rho(d)$$

for any $d \in [d_0, \infty) \setminus D$.

$$\mathcal{M}_d := \left\{ [A] \mid F_A^+ = 0 \text{ and } \|F_A : L^\infty\| \leq d \right\},$$

$$\rho(d) := \sup_{[A] \in \mathcal{M}_d} \left[\lim_{T \to \infty} \frac{1}{8\pi^2 T} \left(\sup_{t \in \mathbb{R}} \int_{\mathbb{S}^3 \times [t, t+T]} |F_A|^2 \, d\mu \right) \right]$$